MTH 1450 Chapter 5 Review

1 Angles and Arcs of Circles

Tool: Angles require a change of thinking from linear coordinates to rotation coordinates. Think protractor instead of ruler. Positive angles are measured counterclockwise. Negative angles are measured clockwise.

Tool: One method of measuring angles is by degrees. There are 360 degrees in a circle. Lesser degree measures indicate some fraction of the circle arc. Greater degree measures indicate multiple wrapping around the circle arc. There are 60 minutes in a degree, 60 seconds in a minute. Degrees/minutes/seconds are used a lot in geography.

Tool: Another method of measuring angles is by radians $\theta = \frac{s}{r}$ using arc length s of circle of radius r. There are 2π radians in a circle, so we measure fractional arc s and compute radian angle . Converting from degrees to radians multiply by $\frac{\pi}{180}$. Converting from radians to degrees multiply by $\frac{180}{\pi}$. Your calculator can convert from radians to degree mode. **Be careful** you know which mode you are in. Radians are used a lot in engineering.

Tool: Angles in standard position. Initial ray is x-axis. The terminal ray determines the angle measure. NOTE: several **coterminal** angles may have the same two rays. You must specify the direction of angle measure from initial ray. **Complementary** angles have measures that add to 90° and **supplementary** angles have measures that add to 180°.

Tool: Arc length (very useful for calculus and engineering). $s = r\theta$ where r is radius and θ is angle measure in **radians**.

Tool: Sector area. $A = \frac{1}{2} r^2 \theta$ or

Tool: Linear speed $v = \frac{s}{T}$

Angular speed $\omega = \frac{\theta}{T}$ (use radian measure)

Using these definitions an $\theta = \frac{s}{r}$ we can relate linear and angular speed using the formula $\omega = \frac{v}{r}$.

Practice Problems

Picturing angles: (see p. 342) Can you draw a picture of positive and negative angle including initial and terminal rays and direction of angle measure?

Degree measures:
Convert 20.16° to degrees, minutes, seconds.
Ans: 20° 9' 36"

Convert: 43° 18' 10" to decimal degrees (3 decimal places)
Ans: 43.303°

Radian measures:
Convert angles of measure 180°, 90°, 45°, 30°, 60° to radians.

Convert angle of measure $\frac{\pi}{5}$ to degrees.

Angles in standard position.
Can you sketch angles of measure 180°, 90°, 45°, 30°, 60°?

Can you give two coterminal angles to $\pi/3$?

What is the complement of a 15° angle? What is the supplement of a 15° angle?

(formula given)Pizza problem: Suppose you have a 10" diameter pizza and a slice that is 60° sector.
What is the arc length of the crust? (remember to convert to radians)
What is the sector area of the crust?

(formula given)Wheel problem: Car speedometers measure speed using angular measurements, but cars travel linearly. If a tire is 0.5 m in radius and the car is traveling at 30 m/s (67 mph) what is the angular velocity in radians per second? Angular velocity in revolutions per minute? If the tire is replaced/overinflated to 0.6 m radius without recalibrating the speedometer, what linear speed corresponds to the angular velocity you just computed?
Ans:
Angular velocity corresponding to 30 m/s is 60 rad/s or 573 rpm.
If radius changed to 0.6 m the new speed is 36 m/s (79 mph) but the speedometer still reads low at 30 m/s.

2 Right Triangle Trigonometry

Tool: Acute angles θ in right triangles have six associated functions defined using the sides of triangle. If we call hypotenuse c, adjacent side b, opposite side a, then $c^2=a^2+b^2$ and

$$\sin(\theta)=\frac{opp}{hyp}=\frac{a}{c}$$
$$\cos(\theta)=\frac{adj}{hyp}=\frac{b}{c}$$
$$\tan(\theta)=\frac{opp}{adj}=\frac{a}{b}$$
$$csc(\theta)=\frac{1}{\sin(\theta)}=\frac{hyp}{opp}=\frac{c}{a}$$
$$sec(\theta)=\frac{1}{\cos(\theta)}=\frac{hyp}{adj}=\frac{c}{b}$$
$$\cot(\theta)=\frac{1}{\tan(\theta)}=\frac{adj}{opp}=\frac{b}{a}$$

Tool: Although the definitions give us insight into what calculators do, for most angles we will use calculators to obtain decimal approximations.

Tool: Recall that inverse functions switch input and output, so inverse trig functions take a ratio and output an angle measure.

Tool: Your calculators also have inverse trig functions $\sin^{-1}, \cos^{-1}, \tan^{-1}$ (also known as arcsin, arccos, arctan) which are **NOT** reciprocals of sin, cos, tan (recall the reciprocal functions were already defined as csc, sec, cot). If you want the inverse functions $sec^{-1}, csc^{-1}, \cot^{-1}$, you need to do more work (see p. 357).

Tool: Given two two sides of a right triangle or a side and an angle, you should be able to solve for the remaining sides and angles by using trig or inverse trig functions.

Evaluating Trig functions. Suppose a right triangle has hypotenuse of 1 and angle of 60°. What's the measure of the other angle? Can you compute sin(60°), cos(60°), tan(60°), csc(60°), sec(60°), cot(60°) using the definition?

Can you do the same for a right triangle with 45° angle?

Can you do the same for the angle adjacent to side of length 4 in a right triangle with sides of length 3,4,5?

Can you do the same for right triangle with 0° angle?

You should be able to evaluate trig functions 180°, 90°, 60°, 45°, 30°, 0° and their radian counterparts.

Calculating trig functions.

$\tan 89.9$ (use degree mode) Ans: 572.96

$csc\left(\frac{\pi}{100}\right)$ (use radian mode) Ans: 31.84

Evaluating inverse trig functions (give result in degree and radian measures). Use what you know about triangles.

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\tan^{-1}(1)$

$sec^{-1}(1)$

Calculating inverse trig functions.

$\sin^{-1}(0.1)$ (in radians) Ans: 0.1002

$\sin^{-1}(0.1)$ (in degrees) Ans: 5.739°

$\tan^{-1}(2)$ (in degrees) Ans: 63.43°

$\tan^{-1}(2)$ (in radians) Ans: 1.1071

$sec^{-1}(5)$ (in degrees) Ans: 78.46°

Solving triangles.

Solve the right triangle with hypotenuse 3 with 70° angle.
Ans: complementary angle 20°, leg opposite 70° measures 2.819, adjacent leg to 70° measures 1.026.

Solve the right triangle with legs of length 2 and 5.
Ans: hypotenuse $\sqrt{29}\approx 5.385$, angle opposite side of length 5 measures 68.20°, complementary angle is 21.80°.

Surveying application: (see p. 363, #42)

Ans. $\frac{1540}{\tan(26.5)}-\frac{1540}{\tan(79.3)}\approx 2797.78\,m$

3 General Angle Trigonometry

Tool: If we draw terminal ray of radius r, then we may define trig functions using x and y coordinates.

$$\sin(\theta)=\frac{y}{r}$$

$$\cos(\theta)=\frac{x}{r}$$

$$\tan(\theta)=\frac{y}{x}=\frac{\sin(\theta)}{\cos(\theta)}$$

$$csc(\theta)=\frac{r}{y}=\frac{1}{\sin(\theta)}$$

$$sec(\theta)=\frac{r}{x}=\frac{1}{\cos(\theta)}$$

$$\cot(\theta)=\frac{x}{y}=\frac{\cos(\theta)}{\sin(\theta)}$$

Tool: Reference triangles are used to evaluate trig functions for 180°, 90°, 60°, 45°, 30°, 0° and their coterminal angles. Just drop a perpendicular from the terminal ray to the x-axis and use triangle trig to evaluate.

Tool: Signs of trig functions are determined by quadrant and vice versa. A sketch can give an idea.

Tool: Periodic behavior of trig functions means that if we add a multiple of 360° to an angle, they give the same output, i.e., if $f=\sin, \cos, \tan, csc, sec, \cot$, then $f(\theta+360\cdot n)=f(\theta)$ in degree mode and $f(\theta+2\pi\cdot n)=f(\theta)$ in radian mode.

Evaluate the six trig functions for angles 0°, 90°,180°

Evaluate using reference triangles with r = 1 (assume degree mode) and express answers exactly without decimals. Then check with calculator.

$\sin(-135)$

$\cos(210)$

$\tan(420)$

(optional) Quadrant detection.

If $\cos(\theta)>0$ and $\sin(\theta)<0$ what quadrant are we in?

If $\tan(\theta)<0$ and $csc(\theta)<0$ what quadrant are we in?

Compute by hand and check with calculator (radian mode)

$\sin\left(\frac{\pi}{2}\right)$

$\sin\left(\frac{5\pi}{2}\right)$

$\sin\left(\frac{13\pi}{2}\right)$

by observing coterminal angles.

4 Trig functions of Real Numbers

Tool: The unit circle simplifies definitions of trig functions. If we let t be a real number that is equal to θ in radian measure and the point $P(t)=(x, y)$ the point of intersection of terminal ray of θ with the unit circle, then as long as we don't divide by zero

$$\sin(t)=y$$
$$\cos(t)=x$$
$$\tan(t)=\frac{y}{x}$$
$$csc(t)=\frac{1}{y}$$
$$sec(t)=\frac{1}{x}$$
$$\cot(t)=\frac{x}{y}$$

Tool: Even-odd identities. Sketches show that sine is odd and cosine is even, i.e.,

$$\sin(-t)=-\sin(t)$$
$$\cos(-t)=\cos(t)$$

If we use those identities we can also derive

$$csc(-t)=-csc(t)$$
$$sec(-t)=sec(t)$$
$$\tan(-t)=-\tan(t)$$
$$\cot(-t)=-\cot(t)$$

Tool: Pythagorean identities.
If $\cos(\theta)=x$ and $\sin(\theta)=y$ are on the unit circle, then $x^2+y^2=1$, so $\cos^2(\theta)+\sin^2(\theta)=1$. And if we divide through by $\cos^2(\theta)$, we obtain $1+\tan^2(\theta)=sec^2(\theta)$. If we divide through by $\sin^2(\theta)$ we obtain $\cot^2(\theta)+1=csc^2(\theta)$.

Converting a radical to trig form. (see p. 382) We use Pythagorean identities to rewrite (where a is constant) $\sqrt{a^2-u^2}$ in simpler form $|a\sin(\theta)|$ substituting $u=a\cos(\theta)$. We may also rewrite $\sqrt{a^2+u^2}$ substituting $u=a\tan(\theta)$ to obtain $|a\,sec(\theta)|$.

Evaluating trig functions of t given $P(t)=\left(\frac{5}{13}, -\frac{12}{13}\right)$.

Even-odd identities.
Do you remember any other even or odd functions?
Let $\sin(t)=\frac{\sqrt{3}}{2}$ and $\cos(t)=\frac{-1}{2}$. What is t?
What are $sec(-t), csc(-t), \cot(-t)$?

If $csc(\theta)=\frac{5}{4}$ and $sec(\theta)<0$, find the other 5 trig functions.

(optional)Trig substitution.
Use to simplify $\sqrt{16-x^2}$
Use to simplify $\sqrt{16+x^2}$

5 Graphs of Sine and Cosine Functions

Tool: Sine and cosine functions both have domain $(-\infty, \infty)$ and range $[-1,1]$. This is because the inputs are angles in radians that can wrap around any number of times, while outputs are coordinates on the unit circle with are bounded in absolute value by the radius.

- We can use the evenness of sine and oddness of cosine to help with graphs.
- Also periodicity helps. The values of sine and cosine are the same at integer multiples of the period 2π.
- If n is any integer, then zeros of sin(x) are at $n\pi$, $(\ldots,-2\pi,-\pi,0,\pi,2\pi,3\pi,\ldots)$, peaks of $\sin(x)$ are at $\frac{\pi}{2}+2n\pi$, $(\ldots,-\frac{7\pi}{2},-\frac{3\pi}{2},\frac{\pi}{2},\frac{5\pi}{2},\frac{9\pi}{2})$ and troughs are at $\frac{3\pi}{2}+2n\pi$, $(\ldots,-\frac{9\pi}{2},-\frac{5\pi}{2},-\frac{\pi}{2},\frac{3\pi}{2},\frac{7\pi}{2},\ldots)$.
- Zeros of cosine occur where peaks and troughs of sine occur. Peaks of cosine occur at $2n\pi$ and troughs occur at $(2n+1)\pi$.

Tool: The function $h(x)=a\sin(x)+c$ scales the amplitude of $\sin(x)$ and shifts the graph upward by c. Similarly, the function $h(x)=a\cos(x)+k$ scales and shifts $\cos(x)$.

Tool: The function $h(x)=\sin(bx)$ has a new period equal to $\frac{2\pi}{b}$ in which cycles occur. Similarly $\cos(kx)$ modifies frequency of cycles.

Tool: The function $g(x)=\sin(x-h)$ phase shifts the sine graph right by h. If $h=\frac{-\pi}{2}$ the graph of sine moves left by $\frac{\pi}{2}$ and looks like cosine.

Similarly, $f(x)=\cos(x-h)$ shifts cosine graph right by h. If $h=\frac{\pi}{2}$, then cosine shifts to look like sine, i.e.

$$\sin\left(x+\frac{\pi}{2}\right)=\cos(x)$$

$$\cos\left(x-\frac{\pi}{2}\right)=\sin(x)$$

Graph of sine and cosine. Can you sketch graphs of $\sin(x)$ and $\cos(x)$? Keep an eye for peaks, troughs and zeros.

Vertically Scaling and Shifting graphs. Using the graph $\cos(x)$ as a reference, can you provide a sketch of $2\cos(x)-2$? What are its new peak and trough values and where do they occur?

Scaling frequencies. Sketch graphs of $\sin(x), \sin(2x), \sin(4x)$ and compare.

Sketch graphs of $\cos(x), \cos\left(\frac{1}{2}x\right)$.

Specify the period of each graph as well as indicating coordinates of a couple troughs and zeros.

Transformation practice. Sketch the graph of $\sin(x-\pi)$. What does it look like?

Sketch graph of $\cos(x-\pi)$ what does it look like?

Sketch the graph of

$$-\cos\left(2x-\frac{\pi}{4}\right)+1$$

period:
vertical scaling:
horizontal shift:
vertical shift:
peaks(max points):
troughs(min points):
oscillating axis crossings:

6 Graphs of Other Trig Functions

Tool: If we look at graphs for sine and cosine, the graph of tangent is their ratio. The graph of $\tan(\theta)=\frac{x}{y}$ has period π because adding π to an angle on the unit circle will change (x, y) to $(-x, -y)$, which gives the same value.

- If n is any integer, then zeros (all odd) of $\tan(\theta)$ are at $n\pi$ (same as for $\sin(\theta)$). Asymptotes (all odd) of $\tan(x)$ occur at $\frac{\pi}{2}+n\pi$. NOTE: be careful that you don't confuse $\tan(x)$ with x^3. They look similar, but $\tan(x)$ has asymptotes, but x^3 does not.
- Zeros of cotangent (all odd) occur where asymptotes of tangent occur $\frac{\pi}{2}+n\pi$. Asymptotes (all odd) of cotangent occur $n\pi$. The cotangent curve looks like the tangent only flipped and shifted between different sets of asymptotes.

Tool: We can use graphs of sine and cosine to sketch their reciprocal graphs cosecant and secant. The graph of cosecant has values of -1 and 1 same as sine, but $csc(x)$ never crosses the x-axis. Instead it opens to vertical asymptotes in the same spots as the zeros of $\sin(x)$ (because we can't divide by zero). Similarly, $sec(x)$ has vertical asymptotes at the zeros of the cosine function.

Recall $a\sin(bx-h)+k$ If we transform sine and cosine, their corresponding cosecant and secant will transform as well.

Graphing tangents and cotangents. Find the period, label asymptotes and zeros.

$\tan\left(2x+\frac{\pi}{4}\right)$

$\cot\left(\frac{x}{3}-\frac{\pi}{2}\right)$

Sketch the graphs of the following functions, finding the period, labeling asymptotes, zeros.

$sec(3x)$ (Hint: use corresponding cosine graph)

$2\,csc\left(\frac{x}{2}-\frac{\pi}{6}\right)-1$

period:
vertical scale:
horizontal shift:
vertical shift:

7 Inverse Trig functions

Tool: Since $\sin(x)$ is not 1-1 we must restrict it to a section where it is, such as $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. Then we can reflect its graph about $y=x$ (switch input x and output y) to obtain $\sin^{-1}(x)$. Similarly we can restrict $\cos(x)$ to $[0, \pi]$ and reflect about $y=x$ to obtain $\cos^{-1}(x)$. If we restrict $\tan(x)$ on $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and reflect about $y=x$, we obtain $\tan^{-1}(x)$.

Tool: Inverse trig composition. Inverse properties work for proper restrictions.

$$\sin(\sin^{-1}(x))=x, \textit{if } -1\le x\le 1$$
$$\sin^{-1}(\sin(x))=x, \textit{if } -\frac{\pi}{2}\le x\le \frac{\pi}{2}$$
$$\cos(\cos^{-1}(x))=x, \textit{if } -1\le x\le 1$$
$$\cos^{-1}(\cos(x))=x, \textit{if } 0\le x\le \pi$$
$$\tan(\tan^{-1}(x))=x, \textit{if } -\infty< x< \infty$$
$$\tan^{-1}(\tan(x))=x, \textit{if } -\frac{\pi}{2}< x< \frac{\pi}{2}$$

Tool: Triangles are useful in evaluating inverse trig functions. Set the output of an inverse trig function equal the the angle θ in a triangle. Then use the input of the inverse triangle to set up lengths of sides. You can then evaluate the resulting expression (see p. 429).

Tool: Similar restrictions to those for cosine, sine and tangent may be used to help define inverse functions of secant, cosecant, and cotangent. Just like for $\sin^{-1}$, the csc^{-1} is restricted to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. Just like for $\cos^{-1}$, the sec^{-1} is restricted to $[0,\pi]$.

Inverse trig graphs. Use what you know about sine, cosine, tangent—restrictions, asymptotes, zeros, peaks, troughs, to sketch $\sin^{-1}(x), \cos^{-1}(x), \tan^{-1}(x)$ on separate graphs.
For each provide
domain:
range:
asymptotes:
zeros:
max point:
min point:

Exact values of inverse trig functions.
Use what you know about triangles and restrictions to provide the exact values of the following (remember to use radians).

$$\cos^{-1}(-1)$$
$$\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right)$$
$$\tan^{-1}(\sqrt{3})$$

Rewriting an Inverse Function.
If $y=3+5\sin^{-1}(4x+1)$, solve for x in terms of y. Determine the restrictions on x and y (use what you know about domain and range of $\sin^{-1}$).

Inverse trig composition.

$$\tan\left(\tan^{-1}\left(\frac{5\pi}{4}\right)\right) \quad \cos^{-1}\left(\cos\left(\frac{-\pi}{6}\right)\right)$$

Inverse trig expressions.

$$\cot\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$$

Give exact answer. You may check with a calculator.

Evaluate exactly (no decimal):

$$csc^{-1}(-2)$$
$$sec^{-1}(-\sqrt{2})$$
$$\cot^{-1}(\sqrt{3})$$