MTH 1450 Chapter 4 Review

1 Inverse functions

Tool: Intuitively, inverse functions help us solve equations. For example if we have the equation $x^3=7$, we apply cube root to both sides. The cube root cancels with the cube on x so we can solve $x=\sqrt[3]{7}$.

Tool: Inverse functions are defined by their composition properties, so $g(x)$ is the inverse of $f(x)$ iff $(f\circ g)(x)=f(g(x))=x$ and $(g\circ f)(x)=g(f(x))=x$. If g is the inverse of f, we name it f^{-1} (this does **NOT** mean $\frac{1}{f}$).

Tool: The graphs of f and f^{-1} are reflections about the line $y=x$, which means if you take points on one graph, switch the x, and y coordinates, we obtain points on the other graph. When we switch x and y, we also exchange domains and ranges.
NOTE: not all functions have graphs that are inverses.

Tool: A function f has an inverse that is also a function iff a every horizontal line passes through its graph in at most one point. (If you switch x and y, horizontal line test becomes vertical line test for the inverse graph.
Any function passing the horizontal line test is said to be **one-to-one**, i.e., every output comes from exactly one input.

Tool: Finding inverse given equation $y=f(x)$. Just switch x and y (replace one with the other where you find them). Then solve for the new $y=f^{-1}(x)$.
NOTE: In applications we don't always use x and y. In general we need to solve for input variable so it becomes the new output variable, the former output becomes input.

2 Exponential functions

Tool: Exponential functions appear in the form $y=b^x$. The base, b must be positive. If $b>1$ the function is increasing. If $b<1$, the function is decreasing. Exponential functions have horizontal asymptote $y=0$. Their domain is all real numbers and range is positive reals. You can use the x^y or ^ button on your calculator to compute values or graph.

Tool: Transforming exponential functions. $y=ab^{x-h}+k$ shifts the graph of b^x right by h and up by k. If a is negative, the graph flips.

Practice Problems

Inverses for solving equations: Cube and cube root are inverse pairs. Can you think of any other inverse pairs?
What's the inverse of $y=\frac{1}{x}$?

Verifying inverses algebraically:
Verify that $y=2x-2$ and $y=\frac{1}{2}x+1$ are inverses using the definition.

Graphs of inverses:
Can you see any relation between graphs of cube and cube root? Can you generate one graph from the other using symmetry?

What is inverse graph of the function $y=\sqrt{x-1}$? What are the domains and ranges?

Testing inverse graphs via horizontal line test.
What is the inverse graph of the function $y=x^2+2$? Is it a function?

Finding inverse functions (if possible):
Find the inverse function of $f(x)=2x-2$
Find the inverse function of $f(x)=x^2+2$
Find the inverse function of $f(x)=-3(x-2)^3+1$

Application:
The equation for converting degrees Celsius to Fahrenheit is $F(C)=\frac{9}{5}C+32$. Find the inverse function $C(F)$ for converting from Fahrenheit to Celsius and use it to compute what 90 F is in Celsius.
Ans: $C(F)=\frac{5}{9}F-\frac{160}{9}$ and $C(90)\approx 32.22$

Exponential graphs.
Sketch the graphs of 3^x and $\left(\frac{1}{3}\right)^x$. Do you notice any symmetry? Why do you suppose there might be symmetry?

Sketch the transformed exponentials, indicating asymptotes, domain and range.
$y=-7^{x-2}$
$y=1-\left(\frac{1}{10}\right)^x$

Tool: Exponential functions are one-to-one, i.e., if $b^x = b^y$, then $x = y$. We can use this property to solve equations.

Tool: Compound interest formula is useful for calculating how your investments grow, or how much the credit card companies make off of you, $S = P\left(1 + \frac{r}{n}\right)^{nt}$, where P is initial amount of investment, S is amount of investment at time t (in years). r is annual interest rate (APR) and n is number of compounding periods (accrued interest is raked into the pile yearly ($n=1$), quarterly($n=4$), monthly($n=12$), ...).

Tool: Continuous compounding formula. If we let n become very large (infinite number of compounding periods) in the compound interest formula, the whole formula reduces to $S = Pe^{rt}$, where $e = 2.718281828\ldots$ (there is an e^x button on your calculator that will approximate better than memory).
NOTE: This same formula is used to model population growth rates, discharge of capacitor, radioactive decay, and lots of other stuff.

Solve for x

$9^{x^2-2x} = 9^3$

Compound interest problem:
Suppose \$100,000 is invested in the bank over 5 years at 5% compounded quarterly. What is the final return?
Ans: about \$128,203.72

Suppose you have \$100,000 owed on a credit card over 5 years at 25% compounded quarterly. If by some miracle you don't have late fees, how much will you owe?
Ans: about \$ 336,185.34

Continuous compounding:
Suppose \$100,000 is invested in the bank over 5 years at 5% compounded **continuously**. What is the final return?
Ans: about \$128,402.54

Capacitor discharge:
$V(t) = V_0 e^{-t/RC}$ describes what happens to voltage V during time t when you charge up a capacitor C with current and then use the energy to warm up a resistor R. For $R = 1000\,\Omega$ and $C = 0.001\,F$ and initial voltage $V_0 = 5\,V$. What is the voltage after 1 second? After 5 seconds?
Ans: 1.839 V after 1 second; 0.037 V after 5 seconds

3 Properties of Logarithmic Functions

Tool: Since exponential functions $y = b^x$ are one-to-one, we can solve for input x in terms of output y by invoking inverse operation $x = \log_b y$. Recall with inverse functions the input and output switch places (output becomes input and vice versa). Make sure you identify the base b and don't confuse it with input or output.

Rearrange from exponential to logarithmic form:

$3^4 = 81$

$2^{x-1} = \frac{1}{8}$

$x^t = 10$

Rearrange from logarithmic to exponential form:

$\log_3 \sqrt{3} = \frac{1}{2}$

$\log_{\frac{1}{7}} x = -2$

$\log_x 25 = -2$

$2 + \ln x = t$

Tool: The function $\log_b(x)$ is also one-to-one and forms an inverse pair with b^x. So $\log_b(b^x) = x$ and $b^{\log_b(x)} = x$.

Tool: Your calculator has two log buttons $\log = \log_{10}$ and $\ln = \log_e$. Some logarithms you can evaluate by hand by writing the input as a power of the base.

Evaluating Logarithms

$\log_{\frac{1}{4}} 2$

$\log(10000)$

$e^{\ln 153}$

Tool: Common properties of logarithms may be derived from similar properties for exponents

$$\log_b MN = \log_b M + \log_b N$$

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

$$\log_b M^p = p \log_b M$$

Tool: Change of base formula from base b to base c

$\log_b a = \dfrac{\log_c a}{\log_c b}$. Normally this is used to convert to log keys on your calculator.

4 Graphing Logarithmic Functions

Tool: Graph of $y = \log_b x$ is obtained by reflecting $y = b^x$ about the line $y = x$. We can use this symmetry and exponential graphs to generate points on log graphs.

Tool: $\log_b(x)$ graph may be (h, k) shifted and reflected via $a \log_b(x-h)+k$

5 Exponential and Log Equations/Inequalities

Tool: Use one-to-one properties. If $b^x = b^y$ then $x = y$.

Tool: If exponential equation can't be written in terms of powers of the same base, apply log or ln to both sides.

Tool: Quadratic equations $a_2 b^{2x} + a_1 b^x + a_0 = 0$ are solvable by substituting $y = b^x$ and factoring or applying quadratic formula to the resulting equation. Then solve for $x = \log_b y$ and check for extraneous solutions.

Write each expression as a sum or difference of multiples of logarithms:

$$\log_3 \frac{5}{x^3}$$

$$\log(x^2 - 4)$$

$$\ln \sqrt[3]{\frac{x^2 (y-1)^5}{z}}$$

Write each expression as a single log:

$$4 \log_7 x - 3 \log_7 y + \log_7 1$$

$$\frac{1}{2} \log(x-1) - \log(y+2) - \log(z)$$

Compute:

$\log_7 0.3$

Ans: about -0.6187

$\log_{\sqrt{2}} \pi$

Ans: about 3.3030

Application:

Acidity is measured by pH which is given by $-\log[H^+]$ where $[H^+]$ is hydrogen ion concentration moles per liter. How much more acidic is a substance with pH 3.2 than a substance with pH 5.4?

Ans: about 158 times

Logarithm graphs:

Can you sketch the graph of $\log_3(x)$ knowing the graph of 3^x? Evaluate points for some small values of x . What happens to the asymptote?

Sketch the graph of $\log_{1/4}(x)$ using the graph of $\left(\frac{1}{4}\right)^x$.

Then sketch the graph $-\log_{1/4}(x+2)-1$ specifying domain, range, and asymptotes.

Solve for t :

$9^t = 3^{t-1}$

Solve for u :

$e^{4u-1} = 5$

Ans: $u = \frac{1}{4}(\ln 5 + 1) \approx 0.6524$

$2^{3u-1} = 3^u$

Ans: $u = \dfrac{\log 2}{3 \log 2 - \log 3} \approx 0.7067$

Solve for x : (a typo on p. 331 says this is not solvable analytically. Actually it is)

$9^x - 3^x - 12 = 0$ Ans: $x = \log_3 4 \approx 1.2619$

Tool: Use one-to-one property of logarithms.
$\log_b x = \log_b y$, then $x = y$.

Tool: Put logarithms together using the properties from section 3 and rewrite in exponential form.

Solve for w :
$\ln(x^2 - x) = \ln(12)$

Solve for x :
$\log_2(x^2 - 9) - \log_2(x + 3) = 3$

Application:
Assuming continuous compounding, how long does it double money at 5% interest rate. At 25% interest rate?
Ans: about 13.86 years at 5%, about 2.77 years at 25%.