MTH 1450 Chapter 3 Review	Practice Problems

1 Quadratic Functions

Tool: Vertex formula. For $y=f(x)=ax^2+bx+c$, we may use completing the square to obtain $y=a(x-h)^2+k$ where the vertex $(h,k)=\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. NOTE: $\frac{-b}{2a}$ appears in the quadratic formula equidistant from the quadratic roots.

For the parabola $x=g(y)=ay^2+by+c$, we obtain the vertex $\left(g\left(\frac{-b}{2a}\right), -\frac{b}{2a}\right)$ (just switch x and y coordinates).

Tool: Quadratic inequalities. Use the vertex formula to draw the parabola. Then choose a test point from each side to mark the appropriate area.

Tool: Maximum and minimum values. For a parabola, the max or min occurs at the vertex. **IMPORTANT:** you can only find maximum for parabola that opens down and minimum for parabola that opens up. Make sure you know which way your parabola opens!

2 Polynomial Functions of Higher Degree

Tool: Power functions $y=x^n$ of integer degree $n\geq 4$ resemble x^2 if n is even and x^3 if n is odd. The difference is higher powers become flatter around the origin. Given that information we can sketch the shifted shapes

$$y=a(x-h)^n+k$$

Tool: Properties of polynomials.

$$y(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0$$

a. Maximum number of turning points for degree n polynomial is $n-1$. For example, if your calculator returns 5th degree graph with more than 4 turning points, it's broken.
b. Maximum number of distinct roots (set $y=0$) for degree n polynomial is n. Roots that are repeated more than once **multiplicity** greater than 1. Roots with even multiplicities are **turning points**(not crossing x-axis). Roots with odd multiplicities are **crossing points** (crossing x-axis). Counting up the roots means you won't cross in the wrong place.
c. **Leading term** behavior. As $|x|$ becomes large, $y(x)$ approaches $a_n x^n$. We factor out highest power and observe that it dominates for high magnitude $|x|$.
d. **Continuity**. You can draw polynomials without lifting pencil from paper. Between any two zeros, the graph goes up and down again or vice versa. Make sure your graph is consistent with leading term and use test points to check.

Vertical parabola:
For $y=x^2-2x-15$. Find the roots and the vertex, then sketch.

Horizontal parabola:
For $x=-2y^2+12y-7$. Find the roots and the vertex, then sketch.

Roots: $3\pm 2.345=5.345, 0.655$

Quadratic inequality:
Solve $y\geq(-2x+3)(2x+5)$ and graph.

Minimax:
For $y=3(x+5)^2$, find the vertex. Is it a maximum or minimum?

Application:
Suppose the revenue function for a shoe store is

$$R(x)=\frac{-1}{5}x^2+20x$$ where x is the price of shoes sold.

Can you find the maximum revenue and the number of pairs that will give it?

Power function graphs:

$y=(x-2)^5+3$

$y=-2x^6-4$

Sketch the graphs indicating leading term, roots (with multiplicities), turning points, and crossing points:

$y=x(x-2)^2$

$y=x^2(x-4)^2$

What is the behavior of each as $|x|$ becomes large?

Application: Suppose a snow cone vendor decides to shrink its giant cones and charge the same price. The formula for snow cone volume is $V=\frac{1}{12}\pi\left(hd^2+d^3\right)$. Where diameter $d=10$ cm and height $h=20$ cm. Two plans exist: decrease h by 2 cm or decrease d by 2 cm. Which do you think the vendor will choose?
Ans: Original 785 cc, shorter 733 cc, thinner 469 cc.

3 Real Zeros (Roots) of Polynomials

Tool: Polynomial theorems.
We can use synthetic division to divide
$f(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0$
by $x-c$. Then by the **Remainder Theorem,** the remainder is $f(c)$. And if that remainder $f(c)=0$, then by the **Factor Theorem**, $x-c$ is a factor of $f(x)$, and c is a zero (x intercept) of $f(x)$. You can use a calculator graph to help locate which factors to try.

Tool: Intermediate value theorem. Because polynomials are continuous, if they take two values at the endpoints of an interval, they must assume all values in between as well. This means if a polynomial changes sign of its value there is a zero crossing. You can use this fact to sketch polynomials or find their zeros to high accuracy using the bisection method.

Tool: graphing calculators can solve
$a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1 x+a_0=0$
using a ROOTS or ZEROS button. In this class you will have problems that you can do with algebra and check with calculator. I encourage you to show steps when possible.

4 Rational functions

Tool: The functions $\frac{1}{x^n}$ are odd if n is an odd integer and even if n is an even integer. The y-axis is their common vertical asymptote and the x-axis is their common horizontal asymptote. Their domain and range both include all numbers but 0.

Tool: Transformation $\frac{a}{(x-h)^n}+k$. The asymptotes of $\frac{1}{x^n}$ intersect at the origin. The asymptote intersection moves to (h,k). The quantity a flips the graph depending on its sign.

Tool: Vertical asymptotes of rational function
$R(x)=\frac{p(x)}{q(x)}=\frac{a_n x^n+\ldots+a_1 x+a_0}{b_m x^m+\ldots+b_1 x+b_0}$ are found by looking at the **poles** (roots of denominator polynomial) $q(x)$ that do not appear in numerator polynomial $p(x)$. After locating a vertical asymptote we use even/odd behavior using the multiplicity of the poles. Test points can only help.
NOTE: If $p(x)$ and $q(x)$ contain a common zero c, we may factor and cancel. The graph does not have a vertical asymptote at c, instead just a hole in the domain.

Sketching a polynomial using factoring:
Factor $f(x)=x^3+5x^2-4x-20$ using synthetic division and sketch.

Sketching a polynomial using peaks and troughs:
Suppose a polynomial $g(x)$ has peaks $g(0)=2$ and $g(4)=2$. It also has troughs $g(2)=-2$ and $g(-2)=-2$. What's the minimum degree of the polynomial. Can we give a sketch of the polynomial with this information? How many zeros does the polynomial have and where are they located?

Application: (see p. 254 #56) Suppose the relation between percent of population x and percent of total income y of a community is modeled by the function $y=3x^4-6x^3+4x^2$. What are the restrictions on x and y? What percent of the total income does bottom 40% of the population account for? (Ans. 33%) What bottom percent of the population accounts for 80% of the income? (Ans. 87%)

Odd/Even Behavior:
Plot some points and sketch graphs of $\frac{1}{x}$ and $\frac{1}{x^2}$ to see where their asymptotes, domains and ranges lie.

Examples:
Sketch $y=2-\frac{1}{(x-1)^2}$

Use long division and then sketch
$y=\frac{-x+3}{x+2}$

Sketch the graphs indicating leading term, poles/zeros (with multiplicities), turning points, crossing points, and odd/even vertical asymptotes:

$y=\frac{1}{9-x^2}$

$y=\frac{x-3}{9-x^2}$

Tool: Behavior of rational function $R(x)=\frac{p(x)}{q(x)}=\frac{a_n x^n+...+a_1 x+a_0}{b_m x^m+...+b_1 x+b_0}$ for large $|x|$ are found by ignoring all terms but largest power of x in numerator and denominator. You then divide the powers of x to obtain the leading term, $(a_n/b_m)x^{n-m}$. If $m>n$, horizontal asymptote is $y=0$. You can use even/odd behavior of $(a_n/b_m)x^{n-m}$ to determine how the function approaches its asymptotes. If $m=n$, the horizontal asymptote is $y=a_n/b_m$. If $m<n$, then behavior for large $|x|$ is like the power function $(a_n/b_m)x^{n-m}$ and again you use even/odd behavior.

Sketch the graphs indicating leading term, poles/zeros (with multiplicities), turning points, crossing points, and odd/even vertical asymptotes:

$$y=\frac{-x^2}{(x+2)(x-1)^2}$$

$$y=\frac{x^2-9}{(4x^2-9)}$$

$$y=\frac{-x^3+5x^2}{x-1}$$

Application: (see p. 268 #39) Suppose cost C in dollars of removing percent pollution y of coal emissions from a certain town is given by the model $C=\frac{100{,}000\,y}{1-2y}$ where $0\le y\le 0.5$. Sketch the graph of C. Analyze cost of pollution control when y approaches 0.5 or 50%.

5 Theory of Equations

Tool: Descartes' rule of signs. Number of positive roots in polynomial $f(x)$ is equal to the number of sign changes or less than that amount by accounting for pairs of complex roots. Number of negative roots is equal to the number of sign changes in $f(-x)$ or less than that amount by an even number representing complex root count.
NOTE: There is no way other than factoring to determine the complex root counts. But factoring is extremely difficult for most polynomials, so we try to gain qualitative information.

Tool: Suppose we **synthetically** divide a real polynomial with positive leading coefficient by $x-c$. If $c>0$ and bottom row of synthetic division is all nonnegative, then c is an upper bound for the real zeros. If $c<0$ and the bottom row alternates sign (0 counts as either sign), then c is a lower bound for the real zeros.
NOTE: Technology can give you the roots at the press of a button, but if you study engineering you have to understand technology (including how to bound roots).

Tool: Once we have found all the real roots of a polynomial, we apply the Conjugate Roots Theorem, which says if $a+bi$ is a root, then so is $a-bi$. We may factor polynomials that are irreducible over the reals or build polynomials using complex roots.

Example:
Suppose a man on the street walks up to you and says that the following polynomial has at least one positive zero:
$x^6+5x^5+6x^4+7x^3+8x^2+9x+10$
Do you believe him?
What's the largest number of negative zeros the polynomial can have?

Factoring polynomials fully:
We know the number of zeros in a polynomial equals its degree when we account for multiplicity. Given that fact, factor fully x^4-16. Does the result agree with Descartes' rule of signs?

Bounding roots of polynomials:
Is $c=2$ an upper bound for the zeros of x^5+x^2-x+1 ?
What about $c=1$?

Is $c=-2$ a lower bound for the zeros of x^5+x^2-x+1 ?
What about $c=-1$?

Building polynomials:
Build a third degree polynomial with real coefficients that has roots $1, 1-i, 1+i$. Do the coefficients agree with Descartes rule of signs? Can you sketch it's graph?

Application: (see example 6 p. 276)
Walk in the footsteps of history. Before 1545, the following problem was considered unsolvable partially because people didn't believe the complex numbers given them by the quadratic formula actually existed.
We know that 49 is the square of 7. The smallest sum of integer factors of 49 is 14. Can you find two numbers that sum to 12 and whose product is 49?