Supplement 1

CONIC SECTIONS

CIRCLES

In Appendix C, page A-17, your text derives the equation of a circle with center C = (h,k) and radius = r as

$$(x-\mathrm{h})^2+(y-\mathrm{k})^2=\mathrm{r}^2 \qquad (1.1)$$

This is often called the **standard or center/radius form of the equation of a circle.**

Example 1. Finding the Standard Equation of a Circle

Find the standard equation of the circle with radius 4 and center C=(4, –1).

Solution: Substituting 4 for r and (4,–1) for (h,k) in equation (1.5) we get

$$(x-4)^2+(y-(-1))^2=4^2$$

or

$$(x-4)^2+(y+1)^2=16$$

Or, squaring the binomials, combining like terms, and making one side equal 0 we obtain the **general form**

$$x^2+y^2-8x+2y+1=0$$

To reverse this process and convert an equation of a circle from its general form to its standard form, we use the method of completing the square. That is, we convert a polynomial of the form $x^2 + \mathrm{b}x$ into a perfect square trinomial by adding

$$\left(\frac{\mathrm{b}}{2}\right)^2$$

to the expression to get

$$x^2+\mathrm{b}x+\left(\frac{\mathrm{b}}{2}\right)^2=\left(x+\frac{\mathrm{b}}{2}\right)^2$$

Example 2. Show that $x^2+y^2+6x-2y-15=0$ is the equation of a circle by converting it into its standard form and then give the center and radius.

Solution: We may proceed as follows:

$(x^2+6x+\quad)+(y^2-2y+\quad)=15$	Isolate variable terms on one side and group x and y terms
$(x^2+6x+9)+(y^2-2y+1)=15+9+1$	Complete the square on each group and balance the equation
$(x+3)^2+(y-1)^2=25$	Write each group as the square of a binomial

From the last equation, we recognize it is the equation of a circle with r=5 and the center is at (-3,1).

PARABOLAS

If a parabola has its vertex at the origin and its axis on either the x or y axis, it is called a **central** conic. The equations of such parabolas are derived in your text on pages 691-692. These equations are:

Equation:	$x^2 = 4\mathrm{p}y$ (2.1)	$y^2 = 4\mathrm{p}x$ (2.2)
Graph opens:	y direction	x direction
p > 0	+y direction (up)	+ x direction (right)
p < 0	– y direction (down)	– x direction (left)
Distance from V to F	$\lvert \mathrm{p} \rvert$	$\lvert \mathrm{p} \rvert$

If we translate these graphs so the vertex is at (h,k), then equation (2.1) becomes

$$(x-\mathrm{h})^2 = 4\mathrm{p}(y-\mathrm{k}) \qquad (2.3)$$

And, if the graph of equation (2.2) is translated so the vertex is at (h,k), the equation become

$$(y-\mathrm{k})^2 = 4\mathrm{p}(x-\mathrm{h}) \qquad (2.4)$$

Equations (2.3) and (2.4) are the **standard forms** of equations of parabolas.

Example 1. Express $y^2+24x-6y+225=0$ in standard form and find the vertex, focus and directrix.

Solution: We may proceed as follows:

$y^2-6y=-24x-225$	Isolate the variables, keep the constant with the first degree term
$y^2-6y+9=-24x-225+9$	Complete the square, balance the equation
$(y-3)^2=-24x-216$	Write completed square as square of a binomial
$(y-3)^2=-24(x+9)$	Factor out the coefficient of the first degree term

This is in the form of equation (2.4), the x binomial is only first degree, so the parabola opens in the x direction. The negative coefficient (–24) indicates 4p = –24 so p = –6 and the parabola opens in the negative x direction (left). The vertex is at (–9,3). Since $\lvert \mathrm{p} \rvert = 6$ this is the distance from the vertex to the focus. The focus is to the left of the vertex so its coordinates are F = (–9–6,3) = (–15,3). The directrix is to the right of the vertex so its equation is $x = -9 + 6$ or $x = -3$.

Example 2. Express $4x+2y^2+8=0$ in standard form and find the vertex, focus and directrix.

Solution: We may proceed as follows:

$2y^2=-4x-8$	Isolate the variables, keep the constant with the first degree term
$y^2=-2x-4$	Solve for y^2, no square to complete
$(y-0)^2=-2(x+2)$	Factor out the coefficient of the first degree term

Again, this is in the form of equation (2.4). Thus, the vertex is (–2,0), the first degree binomial tells us the parabola opens in the *x* direction, 4p = –2 so p = –½ and the parabola opens in the negative *x* direction (left). Again, the focus is to the left of the vertex so its coordinates are (–2–1/2,0) = (–5/2,0). The directrix is c units on the other side of the vertex so its equation is $x = -2+1/2$ or $x = -3/2$.

ELLIPSES

An ellipse is the set of all points in a plane for which the sum of the distances from two fixed points (called **foci**, the plural of **focus**) is a constant.

The standard equation of an ellipse with center at the origin and major and minor axes on the *x* and *y* axes respectively is derived in your text on pages 692-693. It is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (a>b) \tag{3.1}$$

To find the *y*-intercepts, we proceed as usual (replace *x* by 0 and solve for *y*) giving

$$\frac{0^2}{a^2}+\frac{y^2}{b^2}=1$$

or

$$\frac{y^2}{b^2}=1$$

so

$$y^2=b^2$$

and

$$y=\pm b$$

Similarly, we find the *x*-intercepts by replacing *y* by 0 and solving for *x* giving

$$x=\pm a$$

Thus, <u>c is the distance from the center to a focus, a is the distance from the center to the end of the major axis, and b is the distance from the center to the end of the minor axis</u>. Also, the relation between a, b, and c can be recalled by noting that since the sum of the distances from any point on the ellipse to the foci is k = 2a, this must also be true for the endpoint of the minor axis (0,b). Then, by the symmetry with respect to the *y*-axis, the distance from (0,b) to either focus must be a. Then, as shown in Figure 1, we get a right triangle with sides a, b, and c.

So, $a^2 = b^2 + c^2$

Again, to translate the ellipse so its center is at (h,k) instead of (0,0), replace *x* by *x*–h and *y* by *y*–k giving:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad (3.2)$$

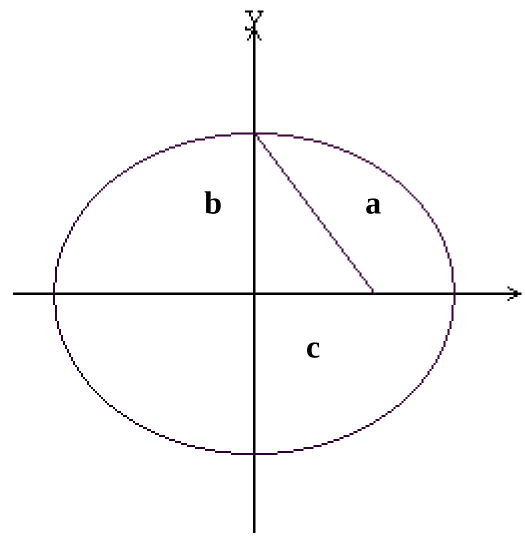

Figure 1

This is the standard form of any equation of an ellipse with center at (h,k) and major axis parallel to the *x*-axis.

If the equation is given in the general form, we employ the technique of completing the squares to convert it into its standard form.

Example 1. Show that $25x^2 + 4y^2 - 100x + 8y + 4 = 0$ is an ellipse by converting it into its standard form.

Solution: We may proceed as follows:

$(25x^2 - 100x) + (4y^2 + 8y) = -4$	Isolate variable terms on one side and group *x* and *y* terms
$25(x^2 - 4x + \ \ ) + 4(y^2 + 2y + \ \ ) = -4$	Factor out the coefficients of the squared terms.
$25(x^2 - 4x + 4) + 4(y^2 + 2y + 1) = -4 + 25 \cdot 4 + 4 \cdot 1$	Complete the square on each group and balance the equation
$25(x-2)^2 + 4(y+1)^2 = 100$	Write each group as the square of a binomial
$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{25} = 1$	Dividing both sides of the equation by 100

The constants a, b and c all give distances from the center with a being the distance to an end of the major axis, b the distance to an end of the minor axis, and c the distance to a focus.

HYPERBOLAS

A hyperbola is the set of all points in a plane for which the difference of the distances from two fixed points (called **foci**, the plural of **focus**) is a constant. The midpoint of the line segment connecting the foci is called the center. The line through the foci is called the transverse axis.

If we let the center be at (0,0), focus $F_1 = (-c,0)$, focus $F_2 = (c,0)$ (the transverse axis is on the *x* axis) and the positive constant k be the constant difference, by the definition any point P will be on the hyperbola if

$$\left\| PF_1 \right| - \left| PF_2 \right\| = k \qquad (4.1)$$

Letting P = (*x*,*y*) and using the distance formula, gives

$$\left| \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} \right| = k \qquad (4.2)$$

Then, letting k = 2a (a >0) and using algebraic manipulations similar to those used for the ellipse we arrive at

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1 \qquad (4.3)$$

Since the difference of two sides of a triangle is always less than the third side, 2a < 2c and a < c. So $a^2 < c^2$ and $c^2 - a^2 > 0$. Letting $b^2 = c^2 - a^2$ and substituting in equation (4.3) we obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (4.4)$$

Notice that if we attempt to find a *y* intercept we get an equation with no real solution. However, if we find the *x* intercepts we get $x = \pm a$. The intersections of the transverse axis with the graph are called vertices (plural of vertex) of the hyperbola. This means a is the distance from the center to a vertex and c is the distance from the center to a focus.

Solving equation (4.4) for *y* gives

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} \qquad (4.5)$$

First note *y* will have a real number value only when $x^2 - a^2 \geq 0$. This occurs when $x \leq -a$ or $x \geq a$. Also notice that the constant a becomes less and less important, relatively speaking, in determining the value of $x^2 - a^2$ as *x* becomes extremely large (or ,as we will eventually denote, as $x \to \infty$). For example, if $x = 3$ while a = 2, $x^2 = 9$ and $x^2 - a^2 = 5$. Here a had a relatively important impact on the size of $x^2 - a^2$. But if $x = 1{,}000$ while a = 2, then $x^2 = 1{,}000{,}000$ while $x^2 - a^2 = 999{,}996$. Here the role of a is almost insignificant. So, as *x* gets very large, $x^2 - a^2$ gets very close to just x^2. Now let us assume that in equation (4.5) *x* is very large (or a negative number which is large in absolute value since the same argument holds). Then we may replace $x^2 - a^2$ by x^2 without significantly changing the eventual value of *y*. That is, the exact value of *y* will be very close to

$$y = \pm \frac{b}{a}\sqrt{x^2} \qquad (4.6)$$

or simply

$$y = \pm\frac{b}{a}x \tag{4.7}$$

This means for very large values of x ($x\to\infty$) or very small values of x ($x\to-\infty$), the graph of the hyperbola with equation (4.4) will get extremely close to the straight lines that are the graphs of equations (4.7). These straight lines are asymptotes of the hyperbola and are useful in sketching its graph.

If we let the center be (0,0) and the foci be c units from the center but on the y-axis (the transverse axis is on the y-axis), the equation becomes

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \tag{4.8}$$

and the asymptotes are

$$y = \pm\frac{a}{b}x \tag{4.9}$$

Distinguishing between equation (4.4) and equation (4.8) is simply a matter of determining which term is positive.

To generalize, let us take the hyperbola whose equation is (4.4) and translate (shift) it so its center is at (h,k) and its transverse axis is parallel to the x-axis. The equation now becomes

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{4.10}$$

and the asymptotes are

$$y - k = \pm\frac{b}{a}(x-h) \tag{4.11}$$

Notice the equations of the asymptotes can be found by replacing 1 in equation (4.10) by 0 and solving for $y-k$.

Similarly, if we translate the graph of equation (4.8) so its center is (h,k) and its transverse is parallel to the y-axis, we obtain

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \tag{4.12}$$

with asymptotes

$$(y-k) = \pm\frac{a}{b}(x-h) \tag{4.13}$$

Again notice that the equations of the asymptotes can be found by replacing 1 in equation (4.12) by 0 and solving for $y-k$.

Equations (4.10) and (4.12) are standard equations of hyperbolas.

Example 1. Convert the equation $16x^2 - 9y^2 - 32x + 36y - 164 = 0$ to its standard form.
Solution. We may proceed as follows:

$\left(16x^2 - 32x\right) - 9y^2 + 36y = 164$	Isolate variable terms on one side and group x and y terms
$16\left(x^2 - 2x + \quad\right) - 9\left(y^2 - 4y + \quad\right) = 164$	Factor out the coefficients of the squared terms.
$16\left(x^2 - 2x + 1\right) - 9\left(y^2 - 4y + 4\right) = 164 + 16 \cdot 1 - 9 \cdot 4$	Complete the square on each group and balance the equation
$16\left(x-1\right)^2 - 9\left(y-2\right)^2 = 144$	Write each group as the square of a binomial
$\frac{\left(x-1\right)^2}{9} - \frac{\left(y-2\right)^2}{16} = 1$	Divide both sides of the equation by 144 and simplify

STANDARD FORMS

We now have the following standard forms of equations of conics:

1. Circle with center at (h,k) and radius r

$$(x-\mathrm{h})^2+(y-\mathrm{k})^2=\mathrm{r}^2 \tag{5.1}$$

2. Parabola with vertex at (h,k) and opening in the *x*-direction.

$$(y-\mathrm{k})^2=4\mathrm{p}(x-\mathrm{h}) \tag{5.2}$$

3. Parabola with vertex at (h,k) and opening in the *y*-direction.

$$(x-\mathrm{h})^2=4\mathrm{p}(y-\mathrm{k}) \tag{5.3}$$

4. Ellipse with center at (h,k) and major axis parallel to the *x*-axis

$$\frac{(x-\mathrm{h})^2}{\mathrm{a}^2}+\frac{(y-\mathrm{k})^2}{\mathrm{b}^2}=1 \qquad (\mathrm{a}>\mathrm{b}) \tag{5.4}$$

5. Ellipse with center at (h,k) and major axis parallel to the *y*-axis

$$\frac{(y-\mathrm{k})^2}{\mathrm{a}^2}+\frac{(x-\mathrm{h})^2}{\mathrm{b}^2}=1 \qquad (\mathrm{a}>\mathrm{b}) \tag{5.5}$$

6. Hyperbola with center (h,k) and transverse axis parallel to the *x*-axis

$$\frac{(x-\mathrm{h})^2}{\mathrm{a}^2}-\frac{(y-\mathrm{k})^2}{\mathrm{b}^2}=1 \tag{5.6}$$

7. Hyperbola with center (h,k) and transverse axis parallel to the *y*-axis

$$\frac{(y-\mathrm{k})^2}{\mathrm{a}^2}-\frac{(x-\mathrm{h})^2}{\mathrm{b}^2}=1 \tag{5.7}$$

GENERAL FORMS

If these standard forms are expanded and simplified, all can be expressed in the **general form**

$$\mathrm{A}x^2+\mathrm{C}y^2+\mathrm{D}x+\mathrm{E}y+\mathrm{F}=0$$

To convert an equation in this general form to one of the standard forms we use some variation of the completing the square technique as illustrated in the above examples. From these, we can reach the following conclusions:

1. If A = 0 or C = 0 but both are not 0, then the equation is of type (5.2) or (5.3) and the graph is, in general, a parabola.

2. If A = C $\neq$ 0, the equation is of type (5.1) and the graph is a circle or one of its degenerate forms.

3. If A $\neq$ C and A and C have the same sign, the equation is of type (5.4) or (5.5) and the graph is, in general, an ellipse.

4. If A $\neq$ C and A and C have different signs, the equation is of type (5.6) or (5.7) and the graph is, in general, a hyperbola.

EXERCISES

In Problems 1 – 8, write each equation in standard form. Identify each graph, also find (when applicable) the center, the vertices, the foci, the equations of asymptotes, and sketch each graph.

PROBLEMS		ANSWERS: Standard forms
1.	$x^2+y^2+8x-6y-24=0$	$(x+4)^2+(y-3)^2=49$
2.	$3x^2+3y^2+12x-18y+12=0$	$(x+2)^2+(y-3)^2=9$
3.	$9x^2+4y^2+54x-8y+49=0$	$\frac{(x+3)^2}{4}+\frac{(y-1)^2}{9}=1$
4.	$4x^2-y^2+16x+4y-4=0$	$\frac{(x+2)^2}{4}-\frac{(y-2)^2}{16}=1$
5.	$y^2-4y-8x-12=0$	$(y-2)^2=8(x+2)$
6.	$x^2-6x+y+5=0$	$(x-3)^2=-(y-4)$
7.	$4y^2+x^2-8y+4x+4=0$	$\frac{(x+2)^2}{4}+\frac{(y-1)^2}{1}=1$
8.	$x^2-4y^2+2x+16y-11=0$	$\frac{(y-2)^2}{1}-\frac{(x+1)^2}{4}=1$